Lecture 5

Metric Characteristics of a Graph

Let $G=(V,E)$ be any connected graph and $u, w \in V$ be arbitrary vertices of G.

Next we construct all paths connecting these vertices u and w:

$$P_k = (u, v_{i_1}, v_{i_2}, \ldots, v_{i_k}, w),$$

$$k = 1, 2, \ldots, r.$$

<u>Def.</u> The distance between two vertices u and w is the length of the shortest path connecting these vertices:

$$\rho(u, w) = \min_{1 \le k \le r} |P_k(u, \dots, w)|.$$

This function has all main properties of metrics:

1. $\rho(u, w) \ge 0,$

   $\rho(u, w) = 0 \iff u = w$

2. $\rho(u, w) = \rho(w, u), \quad \forall u, w \in V$

3. $\rho(u, v) + \rho(v, w) \ge \rho(u, w)$

   $$\forall u, v, w \in V$$

This triangle inequality can be written also as:

$$\rho(u, w) \le \rho(u, v) + \rho(v, w).$$

Prove this inequality.

Graph operations

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be arbitrary graphs.

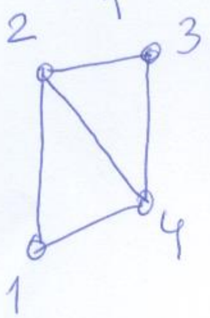
Def. Union $G_1 \cup G_2$ of graphs G_1 and G_2 is defined as

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$$

Def. Intersection $G_1 \cap G_2$ of graphs G_1 and G_2 is defined as

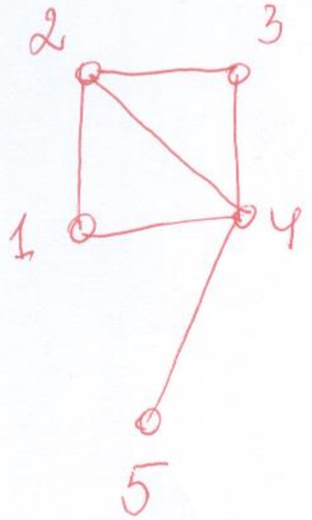
$$G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$$

G_1



G_2

$G_1 \cup G_2$



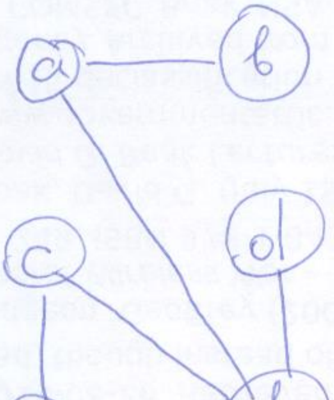
$G_1 \cap G_2$

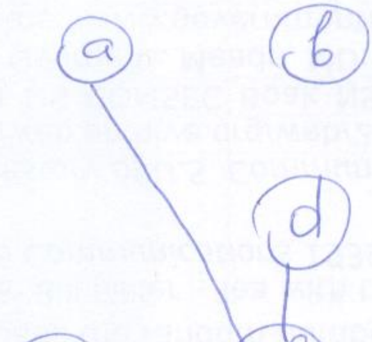
Def. Graph $G' = (V', E')$ is a subgraph of graph $G = (V, E)$ if $V' \subset V$ and $E' \subset E$.

Then we write $G' \subset G$.

Example

G $\qquad\qquad$ $G' \subset G$





Find V and E, V' and E'

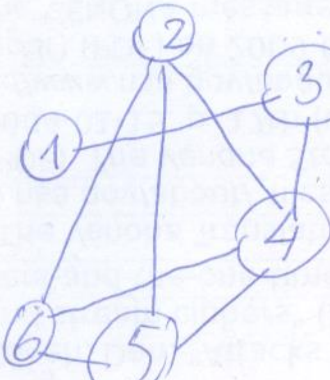
Test if $V' \subset V$, $E' \subset E$

Def. Graph $\bar{G} = (V, \bar{E})$ is called the complement of graph $G = (V, E)$ if $\bar{E}$ is a relative complement of E.

(i.e. $E \cup \bar{E} = \tilde{E}_n$ - a set of all available edges, $\frac{n(n-1)}{2}$.

Example

G



$\bar{G}$

?

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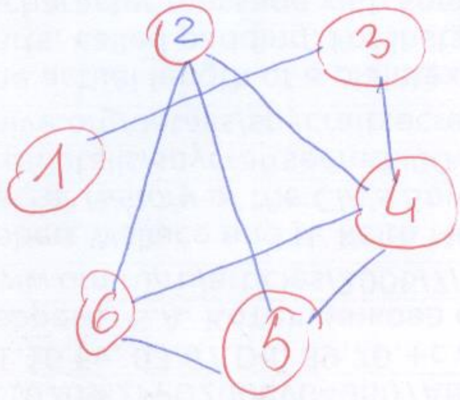
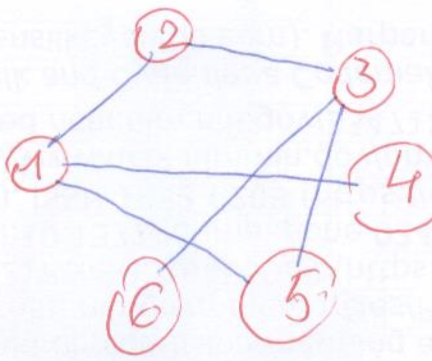
$G=(V,E)$



Complement

$\bar{G}=(V,\bar{E})$



Vertex removal

The removal of vertex v from graph $G=(V,E)$ is denoted by

$$G-v=(V',E')$$

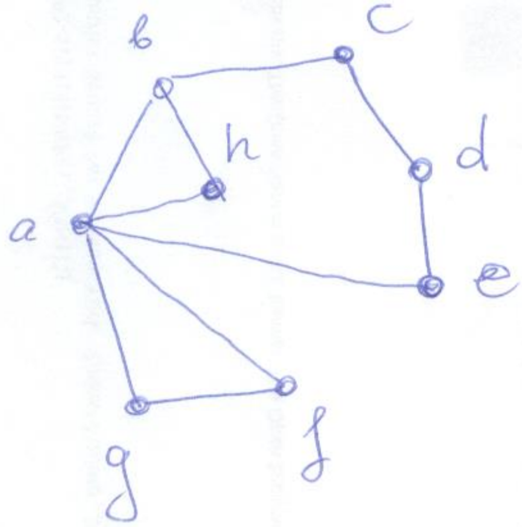
where

$$V'=V\setminus\{v\},$$

$$E'=E\setminus\bigcup_{w\in V:\ \{w,v\}\in E}\{w,v\}$$

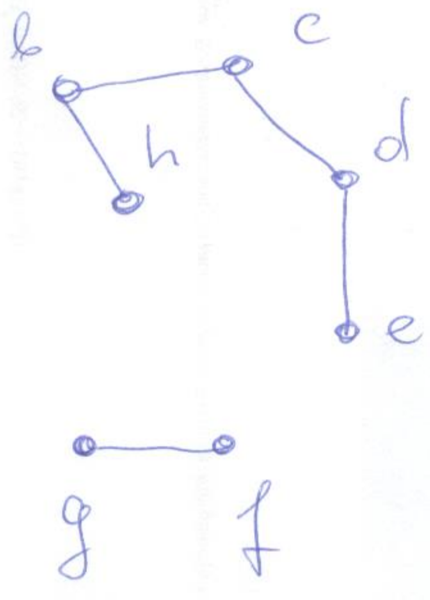
We remove vertex v from set V and all graph edges, which are incident to v.

Example



$G = (V, E)$

Remove graph vertex a.



Edge removal

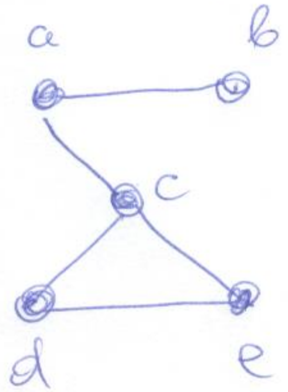
If $\{v, w\} \in E$ is an edge of graph $G = (V, E)$, then

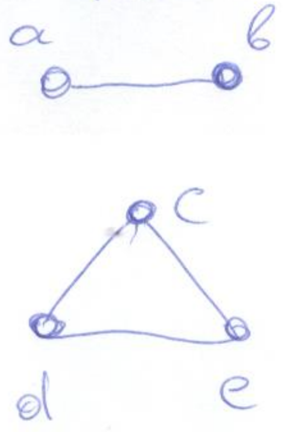
$$G - \{v, w\} = (V, E \setminus \{\{v, w\}\})$$

Remark. We remove an edge, but all incident vertices remain in the graph.

Example.

Remove $\{a, c\} \in E$





Def. The neighborhood of vertex $v \in V$ in graph $G = (V, E)$ is the set of all adjacent vertices

$$\Gamma(v) = \{ w \in V : \{v, w\} \in E \}$$

Number $\rho(v) = |\Gamma(v)|$ is called an order of vertex v in graph $G = (V, E)$.

Def. Any $v \in V$ is called an isolated vertex if $\rho(v) = 0$.

A separating set and a cut
(skiriančioji aibė ir kirpis)

An edge is said ~~to~~ be a bridge of a connected graph, if removing it we disconnect this graph.

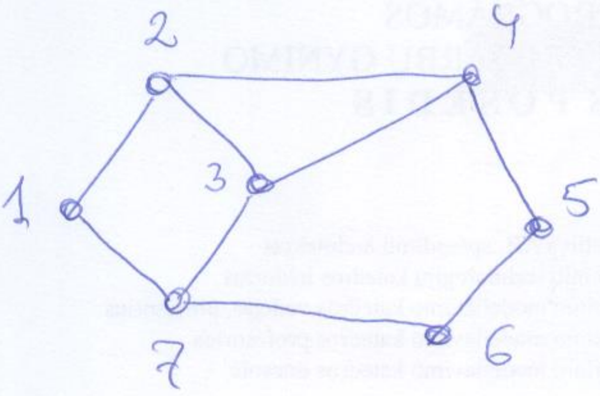
Def 2. If G is a disconnected graph then an edge is said to be a bridge of this graph if it is a bridge of some connected component of this graph

Def. Subset $S \subset E$ is a separating set of connected graph $G = (V, E)$ of graph $G' = (V, E \setminus S)$ is a disconnected graph.

Any minimal separating set is said to be a cut.

Conclusion. Any bridge is a separating set and also a cut.

Example



Graph G.

a) Test that the set of edges

$S = \{\{2,4\},\{2,3\},\{3,4\},\{3,7\},\{6,7\}\}$

is a separating set, but not a cut

b) Set $S_1 = \{\{2,4\},\{3,4\},\{6,7\}\}$

is a cut